Partial Differential Equation Notes
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Based on

1 Introduction and Classifications of PDEs

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu + g = 0$$

for $(x, y) \in \mathbb{R}^2$.

- Hyperbolic: $b^2 - 4ac > 0$
- Elliptic : $b^2 - 4ac < 0$
- Parabolic $b^2 - 4ac = 0$ and $2cd \neq be$ and/or $2ae \neq bd$.

2 Finite Differencing Procedure

U_n is the numerical solution
u_n is the exact solution

1. Define what we want to approximate
2. Replace derivatives with a difference, approximate other terms
3. Apply boundary conditions

3 Local Truncation Error

Global Error: the difference between the numerical and exact

$$e_n = U_n - u(t = n\Delta t)$$

We can use this to define a global error vector for given Δt

$$e_(\Delta t) = [e_0, ..., e_N]^T$$

Assume a derivative of order k is discretised to yield a term with division by exactly k factors of grid spacing,

$$\frac{du}{dt} \to \frac{U_{n+1} - U_n}{\Delta t}$$

Subject to this assumption we define:
Local Truncation Error: the error in the discrete equation when the exact solution is substituted in.
Forward Euler Discretisation:

$$\frac{U_{n+1} - U_n}{\Delta t} = \alpha(n\Delta t)U_n + \beta(n\Delta t)$$

Associated Local Truncation Error:

$$\tau_{n+1} = \frac{u_{n+1} - u_n}{\Delta t} - \alpha(n\Delta t)u_n - \beta(n\Delta t)$$

is the Euler difference equation truncation error.

4 Consistency

Consistent: in a given norm if

$$\lim_{\Delta x \to 0} \|\underline{\tau}(\Delta x)\| = 0$$

where $\underline{\tau}(\Delta x)$ is

5 Stability

6 The Method of Lines

7 Finite Difference for the Heat Equation

8 Method of Lines and Consistency

9 Lax-Richtmyer Stability

Stability Recap: consider

$$\frac{du}{dt} = \lambda u \text{ for } t \in (0, T)$$

Local truncation error satisfies

$$\tau^0 = 0$$
$$\tau^{n+1} = \frac{u^{n+1} - u^n}{\Delta t} - \lambda u^n$$
$$stuff about error$$

$$e^n = -\sum_{p=1}^{n} (1 + \lambda \Delta t)^{n-p} \Delta t \tau^p \text{ for } n \in \{0, ..., N\}$$

blah blah blah we get

$$\begin{aligned}
\|e^n\| &= \|B^n e^0 - \Delta t \sum_{p=1}^{n} B^{n-p} \tau^p\| \\
&\leq \|B^n e^0\| + \Delta t \sum_{p=1}^{n} \|B^{n-p} \tau^p\| \\
\implies \|e^n\| &\leq \|B^n e^0\| + \Delta t n \max_{p \in \{1,...,n\}} \|B^{n-p} \tau^p\| \\
&\leq \|B^n e^0\| + \Delta t N \max_{p \in \{1,...,n\}} \|B^{n-p} \tau^p\| \\
&=
\end{aligned}$$

10 Von Neumann Analysis

von Neumann analysis can be applied to problems which satisfy

- Lax-Richtmyer stability assumptions
- $F^n = 0$
- 1 'spatial' dimension (*)

- Periodic 'spatial' boundary conditions (*)
- Unifrom grid spacing

(*) points can be generalised for some other cases. For $F^n \neq 0$ it is sufficient to prove Lax-Richtmyer stability for the $F^n = 0$ case, since stability is dependent only upon B matrix.

Method

Make the substitution

$$U_m^n = \exp\left(2ik\pi \frac{m\Delta x}{D}\right)$$
$$U_m^{n+1} = \alpha_k U_m^n$$

Find amplification factors α_k (if we can find it for each $k \in \{0, \ldots, M-1\}$ then B must be a normal matrix).
α_k are eigenvalues of B. So choose Δt s.t. all amplification factors are bounded

$$|\alpha_k| \leq 1 \quad \text{for } k \in \{0, \ldots, M-1\}$$

for all positive Δt.

	Forward Euler	Backward Euler	Crank-Nicolson
Matrix solves required?	No	Yes	Yes
Local truncation error	$\mathcal{O}(\Delta t) + \mathcal{O}\left((\Delta x)^2\right)$	$\mathcal{O}(\Delta t) + \mathcal{O}\left((\Delta x)^2\right)$	$\mathcal{O}\left((\Delta t)^2\right) + \mathcal{O}\left((\Delta x)^2\right)$
Condition for stability	$\frac{\kappa \Delta t}{(\Delta x)^2} \leq \frac{1}{2}$	None	None

Identities:

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$
$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

11 Finite Differences for Elliptic PDEs

Poisson Equation:

$$\nabla^2 u = F(x_1, \ldots, x_d)$$

where $\nabla^2 = \sum_{r=1}^{d} \frac{\partial^2}{\partial x_r^2}$

Dirichlet boundary conditions:

$$u = u_D \quad \text{on } \partial\Omega$$

Neumann boundary conditions:

$$\nabla u \cdot \underline{\hat{n}} = g \quad \text{on } \partial\Omega$$

where

$$\int_\Omega F = \int_{\partial\Omega} g$$

and with

$$\int_\Omega u = 0$$

Stencils

Five point Laplacian:

$$\frac{U_{m-1,p}+U_{m+1,p}-4U_{m,p}+U_{m,p-1}+U_{m,p+1}}{(\Delta x)^2} = F(m\Delta x, p\Delta x)$$
$$\text{for } m \in \{1,\dots,M-1\}, p \in \{1,\dots,M-1\}$$

with matrix representation

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} U_{1,1} \\ U_{2,1} \\ U_{1,2} \\ U_{2,2} \end{bmatrix}$$

The Poisson equation can be equipped with Dirichlet conditions on the boundary of the domain, or Neumann conditions provided certain additional constraints are satisfied.

Elliptic problems can be tackled on a computer by using finite difference approximations. There is no "time-like" dimension, so approximations are made on a spatial grid only.

This will lead to a matrix system of equations. There is a straightforward process to follow in order to derive this system.

12 Elliptic Problems and Stability

The Poisson equation in a unit square subject to homogenous Dirichlet boundary conditions

$$u_{xx} + u_{yy} = F(x,y) \text{ for } (x,y) \in (0,1)^2$$
$$u(x=0,y) = u(x=1,y) = u(x,y=0) = u(x,y=1) = 0$$

Discretising

$$\frac{U_{m-1,p}+U_{m+1,p}-4U_{m,p}+U_{m,p-1}+U_{m,p+1}}{(\Delta x)^2} = F(m\Delta x, p\Delta x)$$
$$\text{for } m,p \in \{1,\dots,M-1\}$$
$$U_{0,p} = U_{M,p} = U_{m,0} = U_{m,M} = 0 \quad \text{for } m,p \in \{0,\dots,M\}$$

where $U_{m,p}$ is the discrete solution at $x = m\Delta x$ and $y = p\Delta x$, where $\Delta x = \frac{1}{M}$ with M positive integer.

Notes prove/find stability

13 The Advection Equation

The advection equation:

$$u_t + vu_x = 0$$

14 Characteristic Based Methods

Consider

$$u_t + vu_x = 0$$

where v is a real constant. This has the general solution

$$u(x,t) = Q(x - vt)$$

for some function Q set by the initial and boundary conditions. The solution is constant along lines of constant $x - vt$ (the characteristics).

Characteristic Based Methods: Trace solutions along the characteristics.

One characteristic based method:

1. Introduce fully discrete solution values
2. From a given grid point at "time" level $n+1$, trace backwards along characteristics to "time" level n
3. Interpolate the numerical solution at the origin point of the characteristic at "time" level n

e.g for $v > 0, v\Delta t/\Delta x \leq 1$, linearly interpolating between grid points at $x = a + m\Delta x$ and $x = a + (m-1)\Delta x$

$$U_m^{n+1} = U_m^n + \frac{v\Delta t}{\Delta x}(U_{m-1}^n - U_m^n)$$

Which can be rearranged and found directly via the method of lines, using an "upwind" discretisation in the x-dimension.

More generally can construct a one-sided difference approximation, e.g. in semi-discrete form for $v > 0$

$$\frac{\mathrm{d}U_m}{\mathrm{d}t} + \frac{v}{\Delta x}\sum_{q=0}^{Q}\alpha_q U_{m-q} = 0$$

where the α_s can be cohsen to yield desired properties (e.g. a local truncation error of a given order).

Upwinding Worked Example

Taylor Expansions:

$$\begin{aligned}
u_m^{n+1} &= u_m^n + \Delta t\,(u_t)_m^n + \mathcal{O}\left((\Delta t)^2\right)\\
u_{m\pm1}^{n+1} &= u_m^{n+1} \pm \Delta x\,(u_x)_m^{n+1} + \frac{1}{2}(\Delta x)^2\,(u_{xx})_m^{n+1} \pm \frac{1}{6}(\Delta x)^3\,(u_{xxx})_m^{n+1} + \mathcal{O}\left((\Delta x)^4\right)\\
u_{m-1}^n + u_{m+1}^n &= 2u_m^n + (\Delta x)^2\,(u_{xx})_m^n + \frac{1}{12}(\Delta x)^4\,(u_{xxxx})_m^n + \mathcal{O}\left((\Delta x)^6\right)\\
u_{m-2}^n + u_{m+2}^n &= 2u_m^n + 4(\Delta x)^2\,(u_{xx})_m^n + \frac{4}{3}(\Delta x)^4\,(u_{xxxx})_m^n + \mathcal{O}\left((\Delta x)^6\right)
\end{aligned}$$

$$\begin{aligned}
u_{m-1,p} + u_{m+1,p} &= 2u_{m,p} + (\Delta x)^2\,(u_{xx})_{m,p} + \mathcal{O}\left((\Delta x)^4\right)\\
u_{m,p-1} + u_{m,p+1} &= 2u_{m,p} + (\Delta x)^2\,(u_{yy})_{m,p} + \mathcal{O}\left((\Delta x)^4\right)
\end{aligned}$$

Definitions

- Taylor Series:

- **Consistency:** a numerical method is consistent in a given norm if

$$\lim_{\Delta x\to 0}\|\underline{\tau}(\Delta x)\| = 0$$

where $\underline{\tau}(\Delta x)$ is a local truncation error vector associated with a grid spacing Δx.

- **Convergence:** a numerical method is convergent in a given norm if

$$\lim_{\Delta x \to 0} \|\underline{e}(\Delta x)\| = 0$$

where $\underline{e}(\Delta x)$ is a global error vector associated with a grid spacing Δx.

- **Accurate:** if there is a non-zero real constant C such that

$$\lim_{\Delta x \to 0} \frac{\|\underline{e}(\Delta x)\|}{(\Delta x)^p} = C$$

then the numerical discretisation method is said to be p-th order accurate.

- **Stability:** a numerical method is said to be stable in a given norm if there is some grid spacing Δx_0 st $\forall \Delta x < \Delta x_0$,
  - The discretisation matrix $A(\Delta x)$ is invertible.
  - There is some $C \in (0, \infty)$ st $\forall \Delta x < \Delta x_0$

    $$\|(A(\Delta x))^{-1}\underline{w}\| \leq C\|\underline{w}\|$$

    for all compatible sized vectors $\underline{w}$.

Examples